



FX Singular Spectrum Analysis

Mauricio D. Sacchi*

Signal Analysis and Imaging Group, Department of Physics

University of Alberta, Edmonton, AB, Canada

msacchi@ualberta.ca

Summary

Singular spectrum analysis (SSA) is a method utilized for the analysis of time series arising from dynamical systems. The method is used to capture oscillations from a given time series via the analysis of the eigen-spectra of the so-called trajectory matrix. The trajectory matrix is composed of multiple data views. The singular value decomposition (SVD) of the trajectory matrix can be used for rank reduction and noise elimination. We apply SSA in the FX domain and present a comparison with classical FX deconvolution. The algorithm arising from SSA analysis is equivalent to Cadzow FX noise attenuation, a method recently proposed by Trickett (2008). It is important to stress, however, that Cadzow filtering (Cadzow, 1988) is a general framework for noise reduction of signals and images. Cadzow filtering is equivalent to SSA when considering sinusoidal waveforms immersed in additive random noise. The intention of this abstract is to provide a simple explanation of the basic assumptions made in SSA and its application to the modeling of plane waves.

Introduction

Singular Spectrum Analysis (SSA) is an eigen-spectra decomposition of a 1D time series via the analysis of the trajectory (Hankel matrix) of time series (Vartaud and Ghil, 1989). Geophysicists have used SSA mainly to analyze oscillations of global temperature, precipitation and paleoclimatic series (Ghil et al., 2002).

We will introduce SSA by considering a simple example in the FX domain. Consider a two dimensional signal composed of a single waveform with constant dip. The signal can be represented in the TX and FX domains as follows

$$s(t, x) = w(t - px) \Leftrightarrow S(\omega, x) = W(\omega)e^{-i\omega px}. \quad (1)$$

We now we consider the signal at a fixed frequency ω and assume regularly sampled data: $x = (k - 1)\Delta x$. To avoid notational clutter we drop the dependency on the temporal frequency. The signal in the FX domain is represented by the following harmonic spatial sequence

$$S_n = W e^{-i\alpha n}. \quad (2)$$

We have also introduced the non-dimensional wavenumber $\alpha = \omega p \Delta x$. We can evaluate equation (2) for two adjacent channels and obtain the following recursive expression

$$S_n = P S_{n-1}. \quad (3)$$

The last equation is also the basis for *FX* deconvolution (Ulrych and Sacchi, 2006) and *FX ARMA* filtering (Sacchi and Kuehl, 2000).

The trajectory Matrix and SSA

Equation (3) simple states that the signal is predictable in space. For this particular example, one coefficient P is required to predict the signal. We consider now a signal recorded on $M=7$ equally spaced traces and we form the trajectory matrix

$$\mathbf{M} = \begin{pmatrix} S_1 & S_2 & S_3 & S_4 \\ S_2 & S_3 & S_4 & S_5 \\ S_3 & S_4 & S_5 & S_6 \\ S_4 & S_5 & S_6 & S_7 \end{pmatrix}. \quad (4)$$

In virtue of equation (3) one can rewrite the trajectory matrix as follows

$$\mathbf{M} = \begin{pmatrix} S_1 & P S_1 & P^2 S_1 & P^3 S_1 \\ S_2 & P S_2 & P^2 S_2 & P^3 S_2 \\ S_3 & P S_3 & P^2 S_3 & P^3 S_3 \\ S_4 & P S_4 & P^2 S_4 & P^3 S_4 \end{pmatrix}. \quad (5)$$

Expression (5) clearly shows that the rank of the trajectory matrix for this example is $r=1$. One can show that for *TX* data consisting of L linear events, the *FX* domain data consists of L complex sinusoids and, in addition, the rank of the trajectory matrix is $r=L$. The SVD analysis of (5) can be used to represent the trajectory matrix in reduced-rank form

$$\hat{\mathbf{M}} = \sum_{k=1}^L \lambda_k \mathbf{u}_k \mathbf{v}_k^H. \quad (6)$$

It is important to stress that the absence of noise $\hat{\mathbf{M}} = \mathbf{M}$.

Noise reduction via SSA

FX noise reduction is attained by reducing the rank of the trajectory matrix. In the absence of noise one should be able to recover the data trajectory matrix with the first L singular vectors. In the presence of noise the L strongest singular values and their associated singular vectors span the noise-free signal. One can truncate the SVD reconstruction (6) to obtain an estimate of the noise-reduced trajectory matrix. It is clear that we are not interested in recovering the trajectory matrix but the signal itself. For this purpose we average the filtered trajectory matrix to obtain an estimate of the enhanced signal

$$\hat{\mathbf{s}} = A(\hat{\mathbf{M}}), \tag{7}$$

where the operator A indicates averaging along the anti-diagonals of $\hat{\mathbf{M}}$.

Examples

In the first example we portray SSA filtering of a 1D time series composed of a single oscillation immersed in white noise. Figure 1 shows the signal and the filtered version via SSA. Figure 2 shows the spectrum of singular values. The latter shows two large singular values that are associated to the 2 modes (2 complex exponentials) required to represent a real monochromatic oscillation.

In Figure 3 we compare *FX* deconvolution and *FX* SSA. The data consist of a superposition of plane waves immersed in spatially white noise (SNR=120%). In Figure 4 we display the error panels. It is clear that *FX* SSA (for this particular example) shows less signal leakage than *FX* deconvolution.

Conclusions

This paper presents a simple analysis of SSA filtering in the *FX* domain. It also summaries SSA; a popular method for the analysis of time series. The intention of this paper is to highlight the equivalence of SSA filtering and Cadzow filtering (Cadzow, 1988; Trickett, 2006). The method relies on the assumption of linear events to validate rank reduction as a mean for noise attenuation.

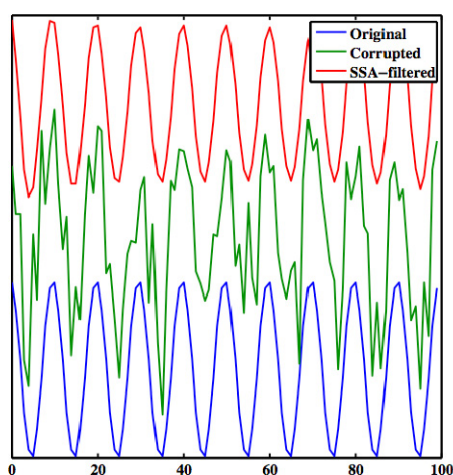


Figure 1. 1D example portraying noise reduction via SSA.

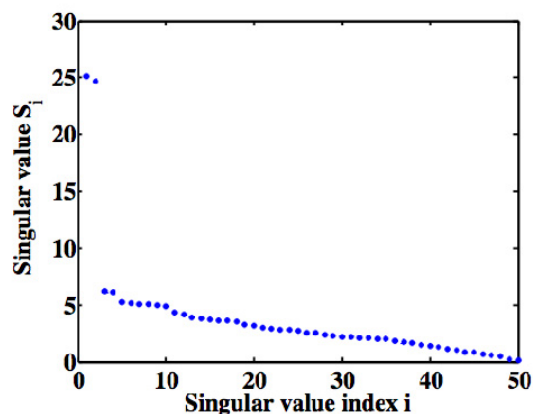


Figure 2. Distribution singular values for the noise corrupted signal in Figure 1.

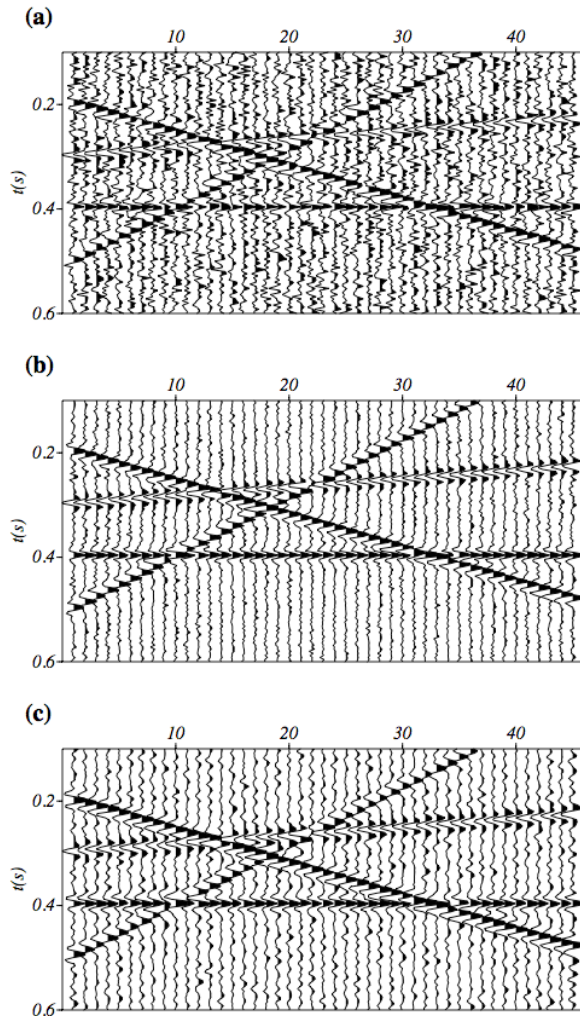


Figure 3. 2D example portraying noise reduction via FX filtering. (a) Input data. (b) FX SSA. (c) FX Deconvolution.

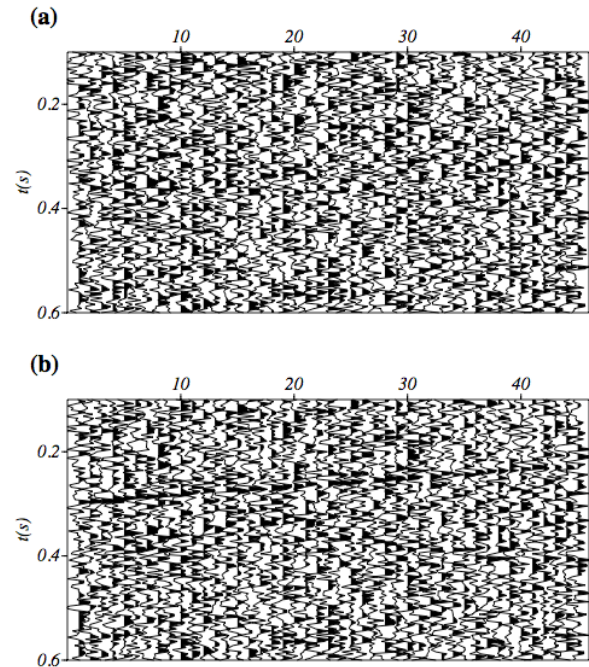


Figure 4. Noise estimates for Figure 3. (a) FX SSA. (b) FX deconvolution.

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