Inverse Modeling the Quality Factor in an Attenuating Medium

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Summary

This work deals with inverse problem of modeling the quality factor for heterogeneous dissipative media. A numerical method for recovering relaxation parameters of the medium is exploited. The relaxation spectrum of a dissipative medium is contained in the spectral measure in the Stieltjes integral representation of the effective complex modulus. The problem of identification of the spectral function from effective measurements of complex viscoelastic modulus in an interval of frequencies has a unique solution, however the problem is ill-posed. To obtain a stable reconstruction of the spectral measure, a discrete approximation of the spectral measure is derived from a rational approximation followed by its partial fractions decomposition. Assuming that the frequency-dependent complex viscoelastic modulus can be recovered from the results of large-scale inversion step in the computation domain or can be modelled for a specific anelastic material. We present a new inversion method based on the constrained rational ($[p,q]$-Pade) approximation of spectral functions with regularization. Numerical examples are given to demonstrate the validity of the algorithm.

Introduction

Attenuating and dispersive effects are often quantified by the quality factor in wave propagation through an anelastic material. Several modeling methods for wave propagation through anelastic media have been presented where the attenuating and dispersive effects are taken into account. In the time-domain, the equation of motion can be written in a form of differential equations by introducing additional memory variables (Day & Minster, 1984; Emmerich & Korn, 1987; Emmerich 1992; Carcione *et al.*, 1988; Ely & Steven, *et al.*, 2008; Xu & McMechan, 1995). The stress-strain relation relates to the material relaxation parameters through these memory variables represented in the complex viscoelastic modulus in the frequency domain. To approximate a quality factor that is approximately constant over a specified frequency band requires 2 to 3 memory variables per decade of bandwidth per stress component per computational unit cell. In order to reduce the cost for computation of synthetic seismograms, we present a new approach which allows us to identify memory variables given measurements of complex viscoelastic modulus, and to further estimate the quality factor.

Modeling of the quality factor and rational approximation for inversion

We consider a homogenized isotropic viscoelastic medium in which the material physical properties are spatially independent. In the frequency domain the relation between stress σ and strain ε in a linear viscoelastic medium is:

$$\sigma(\omega) = M(\omega)\varepsilon(\omega)$$

where $M(\omega)$ is the complex viscoelastic modulus. The quality factor Q as a function of angular frequency ω is defined as

$$Q(\omega) = \frac{\operatorname{Re} M(\omega)}{\operatorname{Im} M(\omega)} = \frac{1}{\tan\theta(\omega)}$$

where θ is the phase of M. $M(\omega)$ is uniquely determined by a given $Q(\omega)$ in a causal medium since $\operatorname{Re} M$ and $\operatorname{Im} M$ must obey a Kramers-Kronig relation. In seismic applications, Q is normally assumed to be frequency-independent or only slowly varying with frequency. The information about the relaxation spectrum of the medium is contained in the spectral measure η in the analytic Stieltjes integral representation of the effective complex modulus (Steven & Mister 1984):

$$F(s) = \frac{M_U - M(s)}{\delta M} = \int_0^\infty \frac{d\eta(x)}{s+x}, \qquad \left(\int_0^\infty \frac{d\eta(x)}{x} = 1, s = i\omega \right)$$

where M_U is the unrelaxed modulus and δM is the relaxation of the modulus. The function $F(s)$ is analytic outside interval $(-\infty,0)$ in the complex s-plane and all its singularities are in the interval $(-\infty,0)$. Based on the theory of inverse homogenization (Cherkaev 2001) and the results (Zhang & Cherkaev 2008), the function $\eta(x)$ is approximated by a step function with a finite number of steps, so that

$$d\eta(x) = \sum_{n=1}^{q} A_n \delta(x - s_n)\,.$$

Here s_n is the n-th simple pole with residues A_n, q is the total number of poles. Thus, the approximation of the dissipation factor $Q(\omega)$ can be written as:

$$Q(\omega) \approx \frac{\operatorname{Re}\{M_U - \delta M \sum A_n/(i\omega - s_n)\}}{\operatorname{Im}\{M_U - \delta M \sum A_n/(i\omega - s_n)\}}\,.$$

A new numerical inversion algorithm for the reconstruction of the spectral measure η is developed using a constrained rational approximation of the spectral function and its partial fraction decomposition (Zhang & Cherkaev, 2008; Zhang & Lamoureux, *et al.*, 2008). We consider $[p,q]$-Pade approximation of the spectral function $F(s)$ in the form of $F(s) \approx F_{[p,q]}(s) = \dfrac{a(s)}{b(s)}$ where $a(s)$ and $b(s)$ are polynomials with order p and q, respectively. The orders p and q of the polynomials in the numerator and denominator of $F_{[p,q]}(s)$ are chosen arbitrarily or could be obtained taking into account the topology of the medium. Assuming that the frequency dependent complex viscoelastic modulus $M(s)$ (or $F(s)$) can be measured or can be modelled analytically. Given the measured data pairs (z_k, d_k) with $d_k = F(z_k)$, the unknown coefficients of polynomials $a(s)$ and $b(s)$ are determined by solving the linear system of equations: $Sc = d$, where the vector c contains all normalized coefficients of two polynomials $a(s)$ and $b(s)$, the vector $d = d_r + \mathrm{i}\, d_i$ and the matrix $S = S_r + \mathrm{i}\, S_i$, subindices r and i indicating the real and imaginary parts of the matrices with entries in terms of data, $\mathrm{i} = \sqrt{-1}$. To derive a stable numerical algorithm, the developed Tikhonov regularized solution c for the above linear system of equations solves the following constrained minimization problem:

$$\min_c \{\| S_r c - d_r \|^2 + \| S_i c - d_i \|^2 + \lambda^2 \| c \|^2\} \quad \text{subject to } -\infty < s_n < 0,\ 0 < \frac{A_n}{|s_n|} < 1, \quad \sum \frac{A_n}{|s_n|} = 1.$$

Here $\|\cdot\|$ denotes the usual Euclidean norm and λ is a positive regularization parameter, A_n and s_n are residues and poles of the partial fraction decomposition for the reconstructed spectral function $F(s)$. The quality factor Q is calculated using the above derived formula.

Results

As a test example we consider inverse modeling dissipation factor Q in a linear viscoelastic medium using measurements of complex frequency-dependent modulus $M(s)$ for the Zener model. The values of material strain relaxation times and stress relaxation times are chosen from (Tal-Ezer & Carcione, Kosloff, 1990) for numerically solving the 1-D viscoelastic equation of motion with L relaxation mechanisms. The derived formula of the spectral function for this analytic model has finite L-terms in a form of partial fractions where the parameters of material relaxation times are represented in the expression of residues and poles of the spectral measure. Using simulated values of the complex modulus $M(s)$ where $s = \mathrm{i}\,\omega$ with $L = 5$ relaxation mechanisms to yield a constant Q = 100 at 50 data points in the seismic exploration band of frequencies from 2Hz to 50Hz, residues and poles of the spectral function shown in Figure 1 and true and computed spectral measure in Figure 2 are reconstructed almost exactly in the case when the order q of $[p,q]$-Pade approximation is bigger or equals to 5. The recovered poles and residues are used to estimate the material strain relaxation times and stress relaxation times with the number of relaxation mechanisms being less than 5 using the constrained $[p,q]$-Pade approximant method (Zhang & Cherkaev, 2008) (Zhang & Lamoureux, *et al*. 2008) with lower order $q < 5$ for evaluating the quality factor Q = 100 shown in Figure 3. The true and computed phase velocity versus frequency is calculated as $c(\omega) = \omega / \mathrm{Re}(k_c)$, where $k_c = \omega / v(\omega)$, the complex velocity $v(\omega) = \sqrt{M(\omega)/\rho}$, the density ρ = 2000kg/m^3, and the relaxed modulus M_R = 8Gpa in the numerical simulations. Our results agree with the published simulations in (Tal-Ezer & Carcione, Kosloff, 1990). The predicted values of relaxation mechanisms can be used for seismic wavefield simulations in viscoelastic media.

Conclusions and future work

A new approach for inverse modeling of the quality factor in an attenuating medium using rational ($[p,q]$-Pade) approximation of spectral functions is presented. The complex modulus of viscoelastic materials contains information about relaxation parameters of the medium. The measurements of frequency-dependent viscoelastic modulus can be used for deriving information about the relaxation mechanisms and modeling the dissipation factor. Further work: The estimated relaxation mechanisms shown in the numerical simulations can be used for simulation of seismic wavefields in viscoelastic media.

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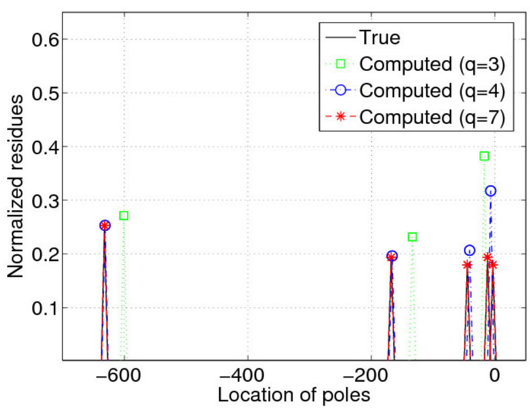

Figure 1 Reconstruction of residues and poles of the spectral function.

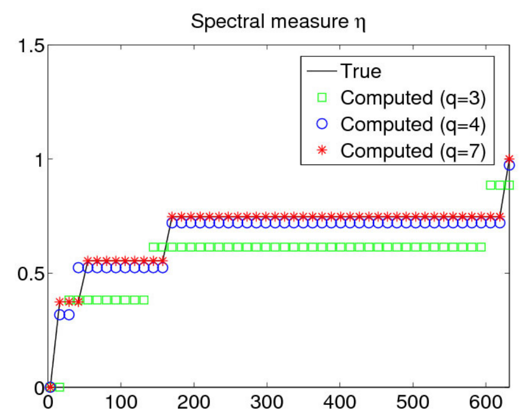

Figure 2 True and computed spectral measure.

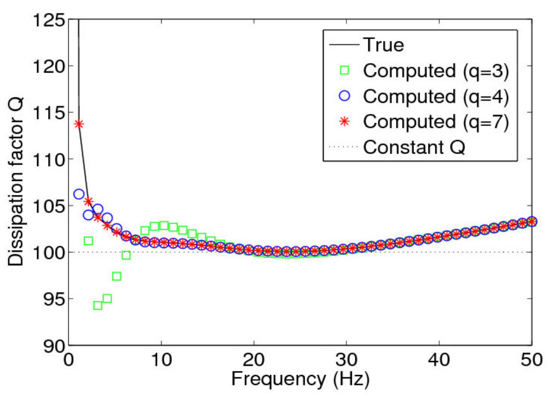

Figure 3 Reconstruction of the quality factor.

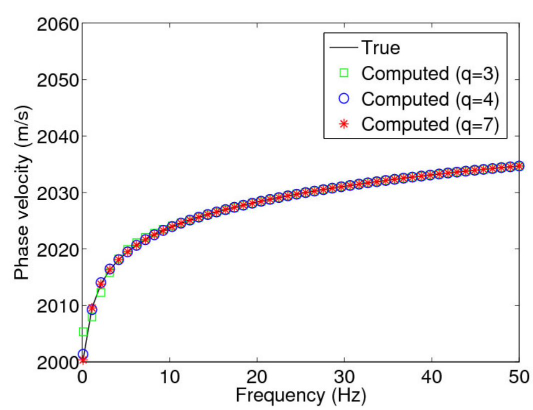

Figure 4 True and computed phase velocity.