A scattering diagram derivation of the eikonal approximation

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Summary

Scattering diagrams are a way of classifying and manipulating the non-linear terms of forward and inverse scattering, ultimately aiding in the derivation of new seismic processing algorithms. The eikonal approximation, a partial solution of the wave equation relevant to seismic processing, may be derived by manipulation of these diagrams. In fact, the diagram derivation achieves its goal in an arguably less roundabout way than other, superficially more elegant, perturbation methods.

Introduction

The eikonal approximation is an expression for modeling scalar wave propagation (Morse & Feshbach, 1953), which, together with its relative the WKBJ approximation, is relevant to seismic exploration (e.g., Clayton & Stolt, 1981; Amundsen et al., 2005). Non-linear scattering theory too is highly relevant to seismic exploration—inverse scattering diagram analysis has led to powerful algorithms for the removal of multiple reflections from seismic data, and is the subject of current research into processing and inversion of primaries (Weglein et al., 2003). Here we link the two, providing a simple derivation of the eikonal approximation from a scattering-diagram analysis of the Born series. We compare it with the derivation of Morse and Feshbach, which is based on a truncation of the integral in the Lippmann-Schwinger equation.

Consider two simple 1D media, an actual medium $c(z)$ and a reference medium c_0, in which actual and reference wavefields, G and G_0 respectively, propagate, from a source depth z_s to an observation depth z_g. G satisfies

$$\left[\frac{d^2}{dz_g^2} + \frac{\omega^2}{c^2(z_g)} \right] G(z_g, z_s) = \delta(z_g - z_s),$$

and G_0 satisfies the same equation but with c_0 replacing $c(z_g)$. Scattering theory is an expression of actual media and fields as series expansions about reference media and fields. Defining $\alpha(z) = 1 - c_0^2/c^2(z)$, the Lippmann-Schwinger equation is

$$G(z_g, z_s) = G_0(z_g, z_s) + \int_{-\infty}^{\infty} G_0(z_g, z')k^2\alpha(z')G(z', z_s)dz',$$

where $k=\omega/c_0$. The Born series then arises through back-substitution of this equation into itself:

$$G(z_g, z_s) = G_0(z_g, z_s) + G_1(z_g, z_s) + G_2(z_g, z_s) + \cdots,$$

where G_n is n'th order in α, for instance, at second order,

$$G_2(z_g, z_s) = \int_{-\infty}^{\infty} dz' G_0(z_g, z')k^2\alpha(z') \int_{-\infty}^{\infty} dz'' G_0(z', z'')k^2\alpha(z'')G_0(z'', z_s),$$

etc. Assuming convergence, summation of a large number of these terms produces an expression for the full wave field. Each term contains propagations and interactions strung together in a chain. For instance, G_2 involves (reading right to left) reference propagation from z_s to z'', where an interaction of strength $k^2\alpha$ occurs, then a further propagation from z'' to z', another interaction, and a final propagation from z' to z_g. The term G_n involves n interactions

and n+1 propagations in the reference medium. To discuss "scattering geometry" is to discuss the characteristic path in z that all or part of G_n takes during its n interactions.
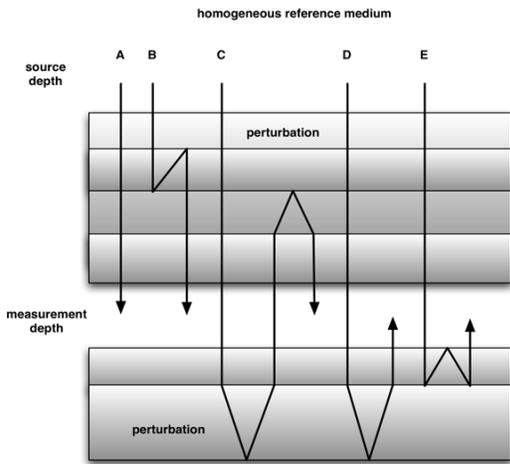

FIG. 1. Even in 1D the wave field in a heterogeneous medium may be quite complex. (A)-(E) are a selection of events in the field G.
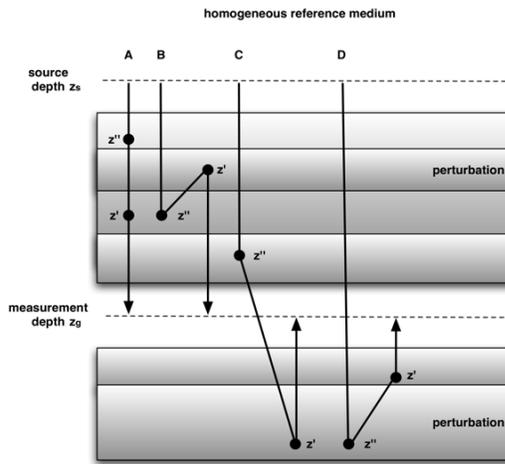
FIG. 2. Evaluating absolute value operations (e.g., at 2nd order) leads to cases (A-D) based on relative scattering geometry.

Scattering diagram derivation of the eikonal approximation

Consider a source plane at z_s embedded in a homogeneous 1D reference medium, above a measurement plane at depth z_g, and assume the reference and actual media to agree at and above z_s, but differ above and below z_g. In Fig. 1 some of the events of the resulting wave field are illustrated. The eikonal approximation is an expression for component (A) in Fig. 1, which dominates when the medium is smooth.   The equations in the introduction reflect this arrangement if z_s is above the depth support of α, and z_g is below or within the perturbation.

Scattering diagrams arise because of the absolute value operation within the reference Green's function (De Santo, 1992):

$$G^0(z_g, z_s) = (i2k)^{-1} e_{i k |z_g - z_s|}.$$

When this is substituted into the Born series terms, and each term is broken up into cases based on the absolute values, each broken up bit has a characteristic scattering geometry. For instance, G_2 decomposes into four cases: (A) $z_g>z'$, $z'>z''$; (B) $z_g>z'$, $z'<z''$; (C) $z_g<z'$, $z'>z''$; and (D) $z_g<z'$, $z'<z''$. These are represented by scattering diagrams (Fig. 2).

Let us allow the geometry of these diagrams to suggest an approach for deriving certain types of wave solution. We know that the eikonal approximation corresponds to the direct part of the wave; no reflections, or changes in direction with respect to the z axis occur as this part of the wave propagates. So, let us see what happens if, instead of summing together all terms in the Born series, we reject from the summation any contribution whose diagram involves a change in direction (Fig. 2 B-D).   At first order, rejecting scattering interactions taking place below z_g leaves a portion of the full wave field we call T_1:

$$T_1(z_g, z_s) = \frac{e^{ik(z_g - z_s)}}{i2k} \left(-\frac{ik}{2} \int_{z_s}^{z_g} \alpha(z')dz' \right), \quad z_g > z_s.$$

At second order (Fig. 2), 3 of the 4 contributing terms involve a change in reference propagation direction. Our program retains only the remaining term, which we will call T_2:

$$T_2(z_g, z_s) = -\frac{e^{ik(z_g-z_s)}}{i2k}\frac{k^2}{4}\int_{z_s}^{z_g}\alpha(z')\int_{z_s}^{z'}\alpha(z'')dz''dz' = \frac{e^{ik(z_g-z_s)}}{i2k}\frac{1}{2}\left(-\frac{ik}{2}\int_{z_s}^{z_g}\alpha(z')dz'\right)^2, \quad z_g > z_s.$$

Repeating this retention/rejection of scattering diagrams over several orders, a pattern is discerned; we use this pattern to collect such terms at all orders. Calling the result T, we have

$$T(z_g, z_s) = \sum_{n=0}^{\infty} T_n(z_g, z_s) = \frac{e^{ik(z_g-z_s)}}{i2k}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{ik}{2}\int_{z_s}^{z_g}\alpha(z')dz'\right)^n, \quad z_g > z_s.$$

This is recognizable as a Maclaurin series. Summing it, we obtain

$$T(z_g, z_s) = (i2k)^{-1}e^{ik\left[z_g-z_s-\frac{1}{2}\int_{z_s}^{z_g}\alpha(z')dz'\right]}, \quad z_g > z_s.$$

This is the eikonal approximation.

Direct integration of the Lippmann-Schwinger equation
We next consider an alternative derivation, that of Morse & Feshbach (1953). Consider again the Lippmann-Schwinger equation, but this time altered, with the integration limit set to the measurement depth z_g. Again calling the field variable satisfying this new equation T, we have

$$T(z_g, z_s) = G_0(z_g, z_s) + \int_{-\infty}^{z_g} G_0(z_g, z')k^2\alpha(z')T(z', z_s)dz',$$

By substituting G_0 into this equation, multiplying through by exp(-ikz_g), and taking the derivative with respect to z_g, a differential equation is obtained:

$$\frac{\partial}{\partial z_g}T(z_g, z_s)e^{-ikz_g} - ikT(z_g, z_s)e^{-ikz_g} = -\frac{k}{2i}e^{-ikz_g}\alpha(z_g)T(z_g, z_s).$$

This may be directly integrated:

$$T(z_g, z_s) = Ce^{ik\left(z_g-\frac{1}{2}\int_{z_s}^{z_g}\alpha(z')dz'\right)},$$

and—importantly for our later discussion—the constant C may be determined as follows. Setting z_g=0, and considering z_s to be slightly negative, so that α=0 between the integral limits, we have that $T(z_g,z_s)|_{z_g=0}$=C. If we further stipulate that the medium is too smooth to reflect any observable wave energy, such that the only contribution to T for this z_g, z_s pair is the direct wave, we may equate this to G_0, and C becomes, under our choices,

$$C = (i2k)^{-1}e^{-ikz_s},$$

at which point the eikonal approximation of the previous section is recovered.

A comparison of the two derivations
At first blush, since the diagram derivation appears to be "throwing away" much more of the wave field than do Morse & Feshbach, the equivalence of the above methods may seem strange. But in fact, the alteration of the integral limit in the first equation of the previous section rejects more of the field than one might expect. In fact, that integral limiting step is *alone* equivalent to the alterations associated with the diagram approach; the determination of C is

completely superfluous, in that it brings no new information to the approximation. Let us see that this is true. Beginning again with the altered Lippmann-Schwinger equation, rather than manipulating it as an integral equation, instead we expand it in series through back-substitution, just as if we were deriving the full Born series. Taking care with the variable z_g, we have:

$$T(z_g, z_s) = G_0(z_g, z_s) + \int_{-\infty}^{z_g} dz' G_0(z_g, z') k^2 \alpha(z')$$
$$\times \left\{ G_0(z', z_s) + \int_{-\infty}^{z'} dz'' G_0(z', z'') k^2 \alpha(z'') \right.$$
$$\left. \times \left[G_0(z'', z_s) + \int_{-\infty}^{z''} dz''' G_0(z'', z''') k^2 \alpha(z''') \left[G_0(z''', z_s) + ... \right] \right] \right\},$$

Examination of this expansion in light of Fig. 2 clarifies that the alteration of the Lippmann-Schwinger equation (as introduced by Morse and Feshbach) interrupts contributions to the wave field precisely where they would begin to incorporate scattering interactions that involve a change of propagation direction: the integral limitation is equivalent to the retention of direct diagrams. Substituting G_0 into the above expansion confirms that the T_i therein are identical to those in the sum on pg. 3.

Conclusions

The point of these derivations and their comparison is to highlight a benefit of the *directness* of the diagram approach. Since their end results are the same, the two approaches appear to differ only procedurally. However, because in the truncated Lippmann-Schwinger integral approach the equation was differentiated, we were forced to additionally argue for the form of C. Since it is based on a series, the scattering diagram derivation lacks a certain expediency, but it clarifies that the *original integral limitation alone* is sufficient to obtain the eikonal approximation in that form. In carrying out the older approach, we now see, it was necessary to at first throw away information critical to the solution, and later return it again by imposing what is in the grand scheme of things a redundant boundary condition. With diagrams we were able to avoid this extra set of arguments. It is worth also emphasizing that making direct waves with a scattering-diagram approach can and has been generalized to multidimensional fields and perturbations (Innanen, 2009), while the truncated Lippmann-Schwinger approach seems to be fundamentally restricted to 1D media. This beneficial directness is common to all diagram-based scattering methods, in forward modeling, and also in inverse scattering seismic processing algorithms.

Acknowledgements

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