

Seismic imaging quantum style

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Summary

One of the advantages of quantum computation is quantum parallelism, which consists in solving many instances of the same problem in a single run. This can be done due to the possibility to set a quantum system in a superposition of states. Although its main limitation is that only one of the states can be read at the end, it is possible to increase the chances of the state we are looking for. We show a framework to pose a seismic imaging problem, usually solved using gradient methods, as a quantum computing algorithm. This algorithm does a global exploration of the model space using quantum parallelism, then it manipulates the quantum phase of their states to increase the chances of reading the model that produces the global minimum. In that way, we can read the model answer at the end of the quantum computation. Something important to notice is that there is no need to compute gradients or Hessians but only to do forward modelling and residual calculation. We illustrate this algorithm with a small instance of a tomography problem. We introduce the basic notions needed to solve the inverse problem and we show a small example step by step to illustrate how the quantum algorithm works. The algorithm has been coded and run in a quantum simulator.

Theory

Classical seismic imaging using optimization methods starts with an initial model that is updated by using the gradient and, possibly, the Hessian of an objective function. Part of the model space is explored near this initial model, as Figure 1, left part, shows. In contrast, seismic imaging quantum style explores the whole model space in parallel, only calculating the residuals, and tagging the model with the smallest residual to increase its probability of being read. Figure 1, right part, shows schematically this.

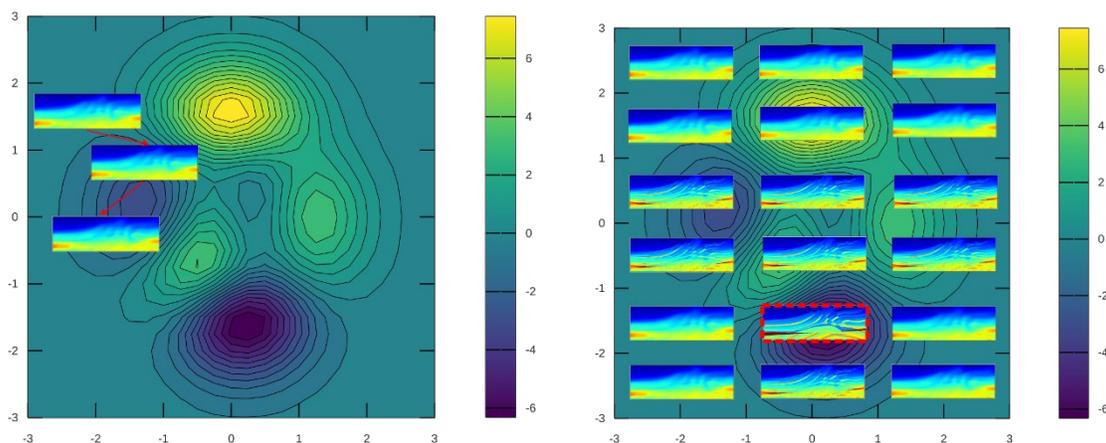


Figure 1. Left shows schematically how a gradient method solves the seismic imaging problem. The arrows are the model updates dictated by the gradient. Right shows the seismic imaging problem solved in quantum style in parallel. The model in red is tagged to increase its probability of being read at the end.

The two main aspects of quantum mechanics for seismic imaging are quantum superposition and quantum measurement. The first one allows us to compute in parallel over all seismic models we can express with the quantum hardware at hand. The second is a limitation because the number of models we can read at the end is only one. But we can overcome it if we tag the seismic model with the smaller residual after modelling to increase its probability of being measured at the algorithm end.

Figure 2 shows the flow diagram of the imaging algorithm quantum style. We start creating the superposition of models, then we calculate the residuals in parallel. The next step is to increase the probability of one model, the one with the smallest residual. We must perform these two last steps several times depending on the size of the model. At the end we read the model with the highest probability that should be the one with the smallest residual.

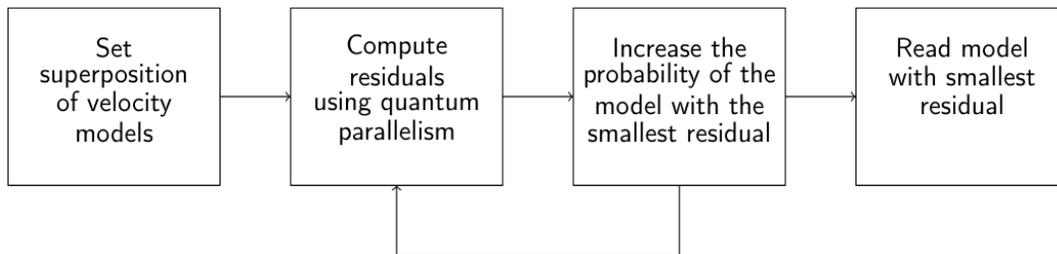


Figure 2. Flow diagram of an imaging algorithm quantum style. It starts by creating a superposition of velocity models. The two middle steps are performed several times to increase the probability of the model with the smallest residual of being read at the last step.

In a classical computer the information is stored in bits and each bit can be in only one of two possible states: 0 or 1. In a quantum computer the quantum bits, called qubits, can be in a superposition of states $|0\rangle$ and $|1\rangle$ (This is the usual quantum notation, and it is read as “ket”). These superpositions are lineal combinations but with complex numbers and normalized so their coefficients squared sum to one. Figure 3 left part shows three classical bits. The shaded portion indicates their state. The left part shows three qubits where the shaded part is how much each qubit is $|0\rangle$ and $|1\rangle$ at the same time. It is usual to express a qubit as $\alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2$ and $|\beta|^2$ are the probabilities of reading the qubit in each respective state.



Figure 3. Left shows three classical bits. The shaded portion indicates if they are 0 or 1. Right shows three qubits. The shaded portion indicates the amount of $|0\rangle$ or $|1\rangle$ that each qubit is.

Before continuing we must see how to group qubits for computations. We know a single qubit has basis elements $|0\rangle$ and $|1\rangle$. We form the basis elements of groups of qubits using the tensor product rules, or basically, concatenating the basis elements of the single qubits in all possible combinations. For example, two qubits have basis elements $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$ and $|1\rangle|1\rangle$.

It is usual to abbreviate the notation by using a single ket symbol around all the qubit values: from $|i\rangle|j\rangle|k\rangle$ to $|ijk\rangle$. A further simplification is to transform the binary values to their decimal representation. For example, $|011\rangle$ would be $|3\rangle$. More of this at the end of this section.

In a classical computer each cell of the seismic model is usually a floating-point number composed of bits. Each cell can store a single value and all the cells store a single velocity model as the top of Figure 4 shows. The bits in the figure form the number r for instance. With a quantum computer each cell is composed of qubits that can be in more than one state at the same time. If we use n qubits per cell, each qubit is $\alpha_i|0\rangle + \beta_i|1\rangle$ and their combination is a superposition of 2^n values in a single cell as the bottom of Figure 4 shows. The shown qubits form the superposition of numbers shown in the figure. If the model has m cells, we can have a superposition of 2^{nm} models.

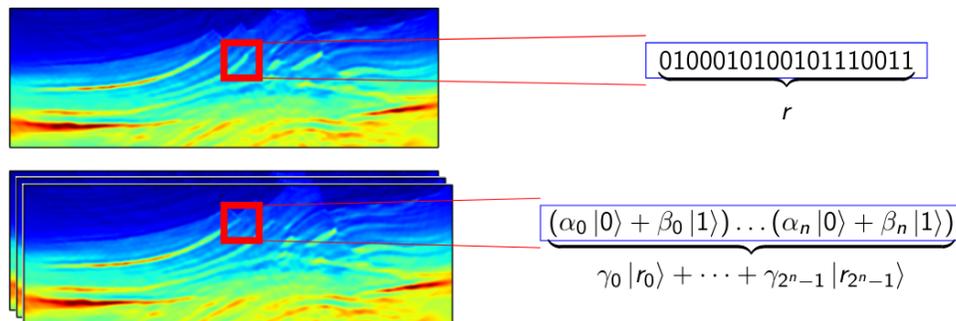


Figure 4. At the top, a seismic model stored in a classical computer has a single value per cell, expressed in bits. At the bottom, each cell of the model is a set of qubits in superposition.

One important property is that we can perform arithmetic operations on a superposition of models and the quantum computer will keep track of and maintain a superposition of results, in a quantum parallel way. The arithmetic operations we are interested in are the ones that calculate the seismic imaging residuals from the model superposition.

The most common way to express quantum computation is with quantum gates. The quantum gates are defined by their effect in the basis values $|0\rangle$ and $|1\rangle$. For example, there is a negation quantum gate, historically denoted by “x” or “not.” This gate transforms $|0\rangle$ to $|1\rangle$ and vice versa, like its classical counterpart. In a superposition it exchanges the coefficients of the superposition:

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{not}} \alpha|1\rangle + \beta|0\rangle \quad (1)$$

There is also an important gate called Hadamard gate, denoted “H”, that does not have a classical counterpart, but helps create the model superpositions we want. It transforms qubit $|0\rangle$ to $1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle$ and $|1\rangle$ to $1/\sqrt{2}|0\rangle - 1/\sqrt{2}|1\rangle$ (The difference is a minus sign in the middle). We will use it to create the initial seismic models’ superposition.

The other important gate is the phase gate, denoted “ph(π)”, which acts in groups of qubits. It changes the sign of the qubits if all of them are $|1\rangle$. For example, it changes $+|11\rangle$ to $-|11\rangle$ but leaves other 2-qubit combinations unchanged. We will use this gate to change the phase of the seismic model with the smallest residual and to increase its probability. The gates just described are not the only ones used in quantum computing (Johnston, 2019), but are the ones necessary for this extended abstract.

The technique to increase the probability of the model with the correct residual is called quantum phase amplification and it is performed with the “mirror” or “Grover’s iteration” circuit. It appears in Figure 5 and is composed of Hadamard, negation and phase gates. It rotates the amplitudes of the coefficients of the superposition around their mean value, so the coefficients below this mean are increased and the ones above are decreased. The example in the next section will show how it works.

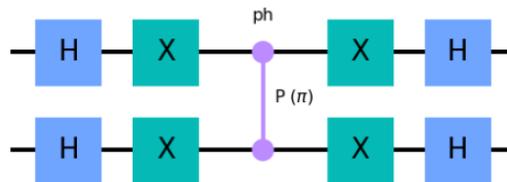


Figure 5. Mirror circuit used in the quantum amplification technique to increase the probability of the seismic model with the correct residual.

We are going to use integer numbers to perform computations. As shown above, a set of qubits can be interpreted as an unsigned integer:

$$|q_{n-1}, q_{n-2}, \dots, q_0\rangle \mapsto \sum_{i=0}^{n-1} q_i 2^i \quad (2)$$

and as a signed integer using two’s complement notation:

$$|q_{n-1}, q_{n-2}, \dots, q_0\rangle \mapsto -q_{n-1} 2^{n-1} + \sum_{i=0}^{n-2} q_i 2^i \quad (3)$$

For example, |101> interpreted as an unsigned integer is $5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$. If interpreted as a signed integer, it is $-3 = -1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$. We will combine unsigned and signed registers in the quantum circuit we use to illustrate the imaging algorithm.

Results

We tested the ideas described before with one of the smallest imaging problems to get a glimpse of the way a quantum algorithm solves an imaging problem. Figure 6 shows this problem instance. It is composed of a single velocity cell and a single source-receiver pair. The source is the star and the receiver is the triangle. Only one traveltimes is inverted. The problem instance we will solve has a raypath of 2 distance units and a measured traveltimes of 4 time units. The answer is 2 slowness units (We refer to slowness as velocity in the following paragraphs).

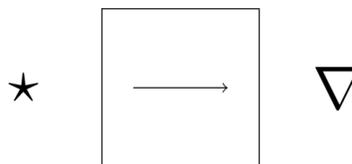


Figure 6. Tomography problem used to illustrate the quantum inversion algorithm. It is composed of a single velocity cell and a single traveltimes between the source, star symbol, and the receiver, triangle symbol.

The main quantum circuit is shown in Figure 7 left part. It consists of horizontal lines for each qubit and boxes that will be explained later. We denote each group of qubits as a quantum

register. The raypath distances are in register $d = [d_1, d_0]$. The superposition of velocity models is in register $v = [v_1, v_0]$. The modelled traveltimes will be placed in register $t^* = [t^*_3, t^*_2, t^*_1, t^*_0]$. The measured traveltimes are input in register $t = [t_3, t_2, t_1, t_0]$. There is a temporary qubit $temp$ and two classical bits out where the output velocity model will be read to at the end. The right part of the same figure contains the quantum circuit states during the setup and computation stages. The columns are the step name and the four quantum registers t , t^* , v and d . We will be referring to each row of this table.

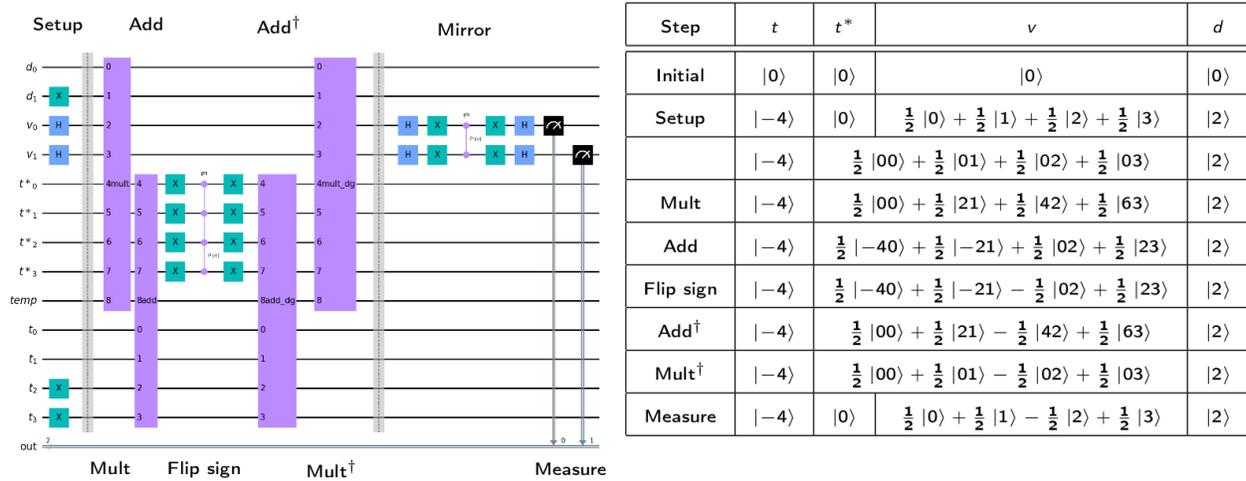


Figure 7. The quantum circuit that solves the tomography problem. On the right is a table showing the evolution of the quantum circuit during the computation.

By convention, all qubits before the setup stage are $|0\rangle$ as step “initial” of the table shows. To input these quantities the setup stage sets $d = |2\rangle = |10\rangle$ and $t = |-4\rangle = |1100\rangle$ using not gates in the appropriate qubits. This stage also sets the velocity register to an equal superposition of all the velocity models by using two Hadamard gates:

$$\begin{aligned}
 |v_1 v_0\rangle &= |00\rangle \xrightarrow{HH} \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \\
 &= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle \\
 &= \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle
 \end{aligned} \tag{4}$$

To ease the explanation, we distribute the register t^* state over the superposition in register v . The $|0\rangle$ in t^* is simply attached in front of every term in v as the result of this distribution.

The next step consists in performing the forward modelling in superposition. The black box labelled “mult” in Figure 7 performs this multiplication between the mentioned registers. Details about the implementation of this quantum arithmetic circuit can be found in Häner et al. (2018). The step mult in the table shows this, for example, this circuit multiplies $|2\rangle$ in d with the 2 in term $|02\rangle$ of combined register t^*v obtaining $|42\rangle$.

The black box with the label “add” in Figure 7 calculates the residuals. It subtracts the measured traveltimes in t from the just calculated ones in t^* and accumulates the result in t^* . Information

about this quantum arithmetic circuit is in Cuccaro et al. (2004). The step add of the table shows the results. For example, -4 is added to the first part of the term $|2\rangle$ resulting in $|-21\rangle$.

The centre of the computation circuit in the same figure is in charge of flipping the phase of the state with the correct answer. In this idealized noise free problem instance the correct model is the one with residual equal to $|0\rangle = |0000\rangle$. Notice that the only the term $|02\rangle$ changes phase because it first part is all $|1\rangle$ after flipping:

$$|02\rangle = |0\rangle|2\rangle = |0000\rangle|2\rangle \xrightarrow{\text{not}^4} |1111\rangle|2\rangle \xrightarrow{\text{pha}(\pi)} -|1111\rangle|2\rangle \xrightarrow{\text{not}^4} -|02\rangle, \quad (5)$$

while the other terms like $|23\rangle$ does not change phase.

Up to this point we have the residuals in the quantum circuit state, but the only essential information is the phase that tags the correct velocity model. In quantum computing it is normal to undo the computation before proceeding further because the different states can be entangled (Johnston, 2019), and this can damage the following stages.

The two boxes labelled “sum_dg” and “mult_dg” are the inverses of the boxes “sum” and “mult.” The table shows this inverse computing. Notice that all states return to their original values but the phase attached to the term with the correct velocity model is preserved.

Now that the state of the circuit has the term with the correct velocity model with a phase different from the other terms, we apply the mirror circuit to the register v. Figure 8 shows this. Before the computation stage, the register v is in an equal superposition of velocity models. After the computation stage the amplitude of term $|2\rangle$ changes to $-1/2$ like in the left part of the figure.

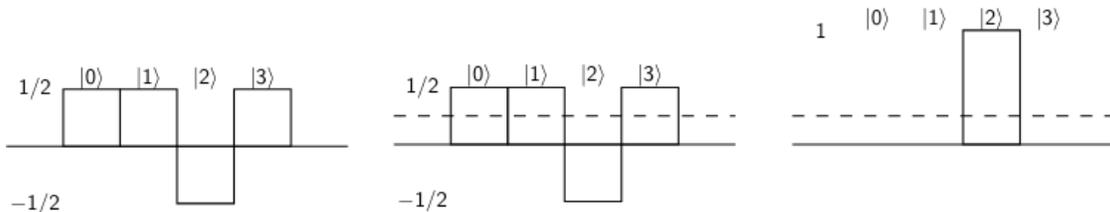


Figure 8. Mirror stage effect on the velocity models superposition in register v. The amplitude of model with the negative phase is increased to 1 so it is certain to be read at the end.

The mean of the amplitudes is $(1/2 + 1/2 - 1/2 + 1/2)/4 = 1/4$. The centre part shows the mean of all the amplitudes as a dashed line. The mirror circuit reflects the amplitudes using the mean as the reflection axis. In this case the amplitude of term $|2\rangle$ becomes 1 while the others vanish. At this point, a reading of the register v would get the velocity model $|2\rangle$ that is the correct answer to this problem instance. It is important to mention that other problem sizes need more iterations to achieve a high probability of the correct seismic model. The next section elaborates more on this.

We coded the quantum algorithm just described in Python using the IBM Qiskit libraries IBM Computing (2021b).

Conclusions

The main advantage of using quantum computation for solving inverse problems is using quantum superposition. With quantum superposition we can calculate the residuals of a set of velocity models in parallel. The main drawback is that it is not possible to read all of them but just one, depending on the coefficients of the quantum superposition. It is necessary to increase the

probability of one term, the one corresponding to the inversion solution, by applying the mirror or Grover's iteration several times. This approach is the same used in quantum searching algorithms (Grover, 1997) and requires around $O(\sqrt{N})$ iterations when N models are simulated in superposition. It is still a huge advantage because it allows us to explore the model space with the same number of iterations of a single gradient-based inversion and obtain the global minimum.

In addition, the actual computation done in superposition is simpler than the classical one because there is no need to compute gradients or Hessians. Only forward modelling and residual calculation are needed, and this simplifies the quantum operations.

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