Learning to solve elastic wave equation with the Clifford Fourier neural operator

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Summary

Neural operators are extensions of neural networks which in supervised training learn how to map complex relationships, such as classes of PDE. Recent literature reports efforts to develop one type of these, the Fourier Neural Operator (FNO), such that it learns to create relatively general solutions to PDEs such as the Navier-Stokes equation. Clifford algebra is very useful for representing multidimensional data. In this study, we use the Clifford Fourier Neural Operator (CFNO) be trained to learn the elastic wave equation from a synthetic training data set. CFNO attempts to find a manifold for elastic wave propagation. On that manifold, wave fields are represented in lower dimensions than those needed for standard solutions, and the calculations for wave propagation are correspondingly simpler. The CFNO combines a linear Clifford fully connected transform, the Clifford Fourier transform, and a non-linear local activation to produce a network with sufficient freedom to map from a general parameterization of a forward wave problem to its solution. Post-training, the CFNO is observed to generate accurate elastic wave fields.

Theory

Clifford algebras serve as a fundamental nexus where geometry melds with algebra, originally conceived to streamline the spatial and geometric interrelationships spanning a multitude of mathematical entities. These algebras seamlessly integrate various mathematical structures, including real numbers, vectors, complex numbers, quaternions, and exterior algebras. Remarkably, Clifford algebras extend beyond standard vector analysis's traditional scalar and vector elements by incorporating spatial elements that encapsulate planes and volumes. Consider the vector cross product in three-dimensional space, effectively represented in Clifford algebras as a plane segment delineated by the two vectors involved. Although represented by three unique components akin to a vector, the cross product distinguishes itself by its characteristic sign reversal upon reflection, which is not a property of true vectors. Clifford algebras encapsulate these diverse spatial elements into entities known as multivectors. In the current research, we supplant the standard feature field operations found in deep learning frameworks with operations derived from Clifford algebras that act on fields of multivectors. The principles of Clifford algebras govern the operational dynamics and interrelations within multivectors. To illustrate, convolutional kernels are endowed with multivector elements to facilitate convolution over multivector-valued feature maps.

The Fourier Neural Operator (FNO) represents a novel approach to learning operators that map between infinite-dimensional function spaces. This methodology is particularly significant in solving partial differential equations (PDEs) where traditional neural networks face challenges due to the high dimensionality and complexity of the function spaces involved. The advent of deep learning has provided powerful tools for function approximation in high-dimensional spaces. However, the application to PDEs has been limited by the need for architectures that can inherently handle the mapping of functions. The Fourier Neural Operator (FNO) is designed to address this by leveraging the expressive power of neural networks in the Fourier domain.

This study integrates Clifford algebra with the Fourier neural operator, yielding the Clifford Fourier Neural Operator (Clifford FNO). The aim is to train the Clifford FNO to adeptly solve the isotropic elastic wave equation, thereby generating the partial derivative wavefields denoted as $\partial_x V_x$, $\partial_z V_z$, $\partial_z V_x$, and $\partial_x V_z$. We evaluate the efficacy of two training methodologies: the rollout and one-step methods. The findings indicate that the rollout method outperforms the one-step method in generating more accurate wavefields, despite the latter initially seeming more congruent with our expectations of a waveform operator. This report is structured as follows: Firstly, a concise overview of Clifford algebra is provided, detailing the fundamental calculation principles inherent to the algebra, which is mostly cited from Brandstetter et al. (2022). Subsequently, the report delves into an exposition of the Clifford Fourier Operator, highlighting it as the pivotal distinction from the conventional FNO. Lastly, we expound upon the training methodologies employed and present the training outcome. The details about the Clifford FNO operator are complicated to explain in detail, one can find the detail instruction in Brandstetter et al. (2022). In this abstract, we will just demonstrate the calculations in Figure 1.

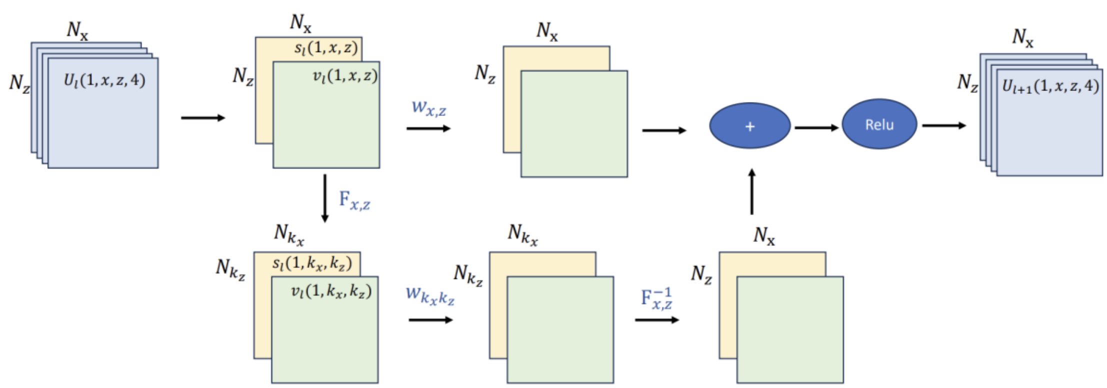

Fig. 1. The architecture of the Clifford Fourier neural operator, using one Clifford multivector U_l, with a dimension of $C^{1\times N_z\times N_x\times 4}$ for the input as an example. The Clifford multivector U_l is firstly being viewed as dual parts, which means that it will be changed into the spinor part s_l, , and vector part v_l,. The complex-valued spinor part s_l, and vector part v_l,, each of which has the dimension of $C^{1\times N_z\times N_x\times 1}$, will undergo the 1D convolution, and the Fourier domain multiplication. Finally the spinor part s_l,and vector part v,are transformed back into the Clifford multivector with dimension $C^{1\times N_z\times N_x\times 4}$, which will be regarded as the updated data U_{l+1}.

Results

The one-step training methodology is adopted for the training process. The demonstration of the training method is presented in Figure 2. In the training stage of the one-step method, the network uses several time steps of the wavefield as input (3-time steps in this Figure) and is trained to generate the wavefield of one-time step. In the validation stage of the one-step training, the network uses several time steps of the wavefield as input (3-time steps in this Figure) and is trained to generate the wavefield of one-time step, but the wavefields generated by the neural network will be used as the input for generating the next time step. A demonstration of the validation stage of one step training is plotted in Figure 3, and more details about the one-step training can be found in Wei and Fu (2022).

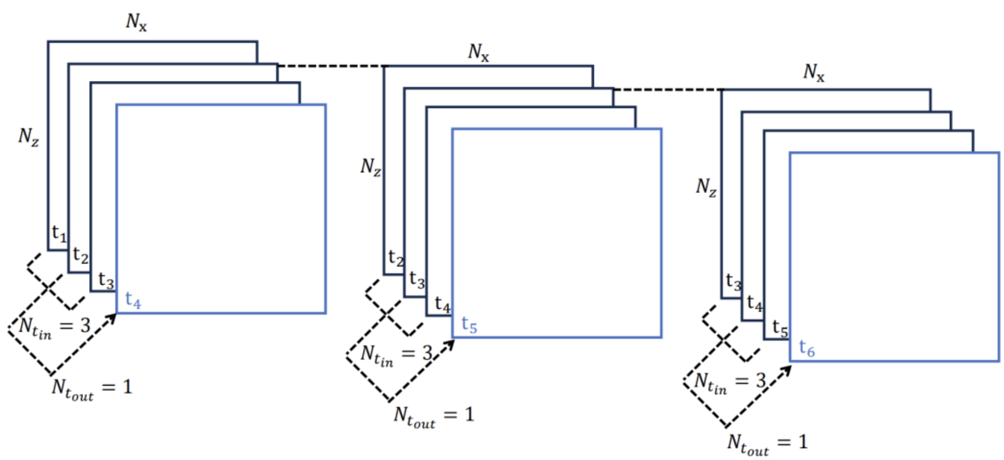

Fig2. One-step training method training stage. In the training stage of the one-step method, the network uses several time steps of the wavefield as input (3-time steps in this Figure) and is trained to generate the wavefield of the next one time step.

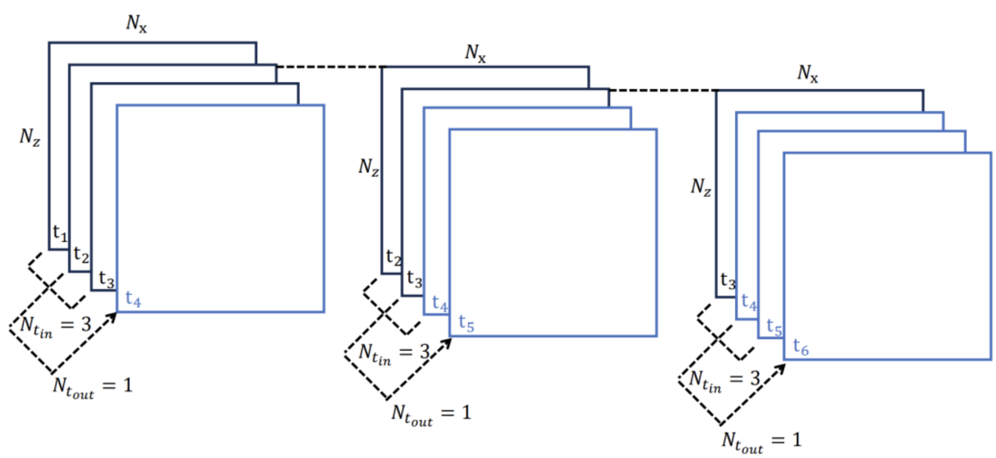

Fig3. One-step training method validation stage. In the validation stage of the one-step training, the network uses several time steps of the wavefield as input (3-time steps in this Figure) and is trained to generate the wavefield of one time step, but the wavefields generated by the neural network will be used as the input for generating the next time step.

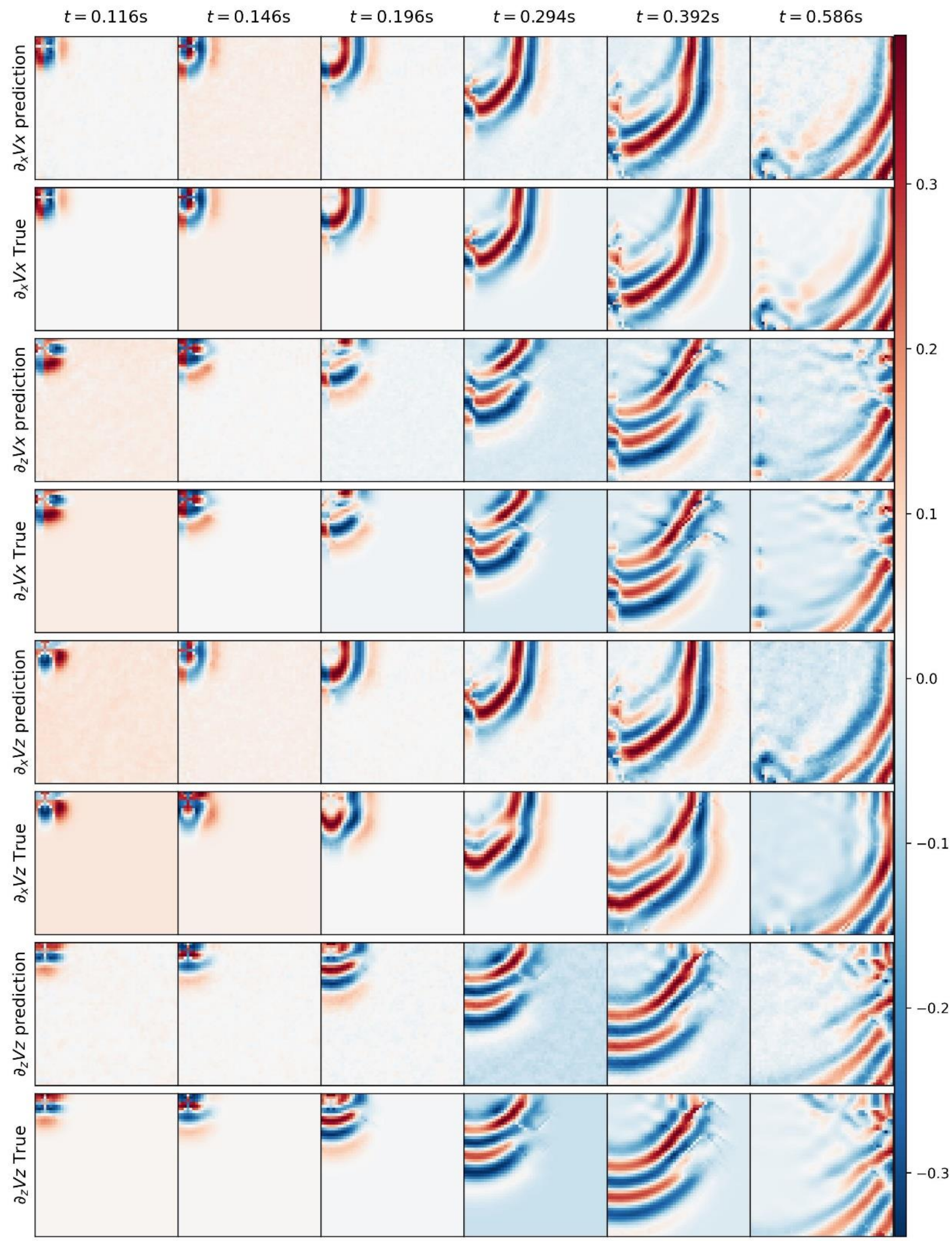

Fig. 5. A example of the snapshots of the wavefields at six-time steps, $\partial_x V_x$, $\partial_z V_z$, $\partial_z V_x$, and $\partial_x V_{z^z}$, generated by the Clifford FNO. The Clifford FNO is trained by using the one-step training methodology.

Figure 5 represents the efficacy of a Clifford Fourier Neural Operator (FNO) that has been trained using the one-step training methodology to predict wavefields. This figure displays a series of snapshots at six time steps: t = 0.002s, 0.004s, 0.008s, 0.01s, 0.012s, and t = 0.02s. The predictions and true values are shown for the partial derivatives of the velocity field components: **,** $\partial_x V_x$, $\partial_z V_z$, $\partial_z V_x$ **, and** $\partial_x V_z$**.** The one-step training approach focuses on training the network by feeding it single steps of the wavefield at a time, a departure from training on entire sequences. This method has implications for the neural network's ability to predict immediate future states with higher precision. In the initial time steps (t = 0.002s to t = 0.008s), the Clifford FNO shows a high degree of accuracy, with the predicted wavefields closely matching the true data in both pattern formation and intensity levels.

The clear definition of wavefronts suggests that the network has successfully captured the essential dynamics of the physical system at these early stages. As time advances to t = 0.01s and beyond, subtle discrepancies begin to emerge. Specifically, the predictions show slight deviations in wavefront sharpness and positioning, indicating a slight model-performance degradation as the pre- diction horizon extends. At the final time step shown (t = 0.02s), although the network still captures the general direction and behaviour of the wavefields, there is a noticeable difference in the intensity and complexity of the wave patterns. The predicted wavefields appear smoother and less nuanced compared to the intricate patterns present in the true data.This analysis suggests that while the Clifford FNO is capable of capturing the immediate dynamics of the wavefields with high fidelity, its performance diminishes when predicting further into the future. This could be due to the limited temporal context provided by the one-step training approach. Training on longer sequences or employing a multi-step training strategy might provide the network with a more comprehensive understanding of the temporal evolution of the wavefields, potentially improving long-term prediction accuracy.

Conclusion

In this study, we train a fast-forward modeling method with the Cliffrod Fourier Neural Operator(FNO). The network consists of three parts, which are two dimensions projection layers that operate on time and several Fourier layer that learns the spatial partial derivaives. The power of the Fourier Neural operator comes from the combination of the Clifford linear operation, operators that resemble partial differential calculation (via the Clifford Fourier transform), and the non-linear local activation. The numerical tests suggest that FNO could generate promising wavefields within certain prediction steps, however, with decreasing accuracy as time propagates.

Acknowledgements

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References

[1] Brandstetter, J., Berg, R. v. d., Welling, M., and Gupta, J. K., 2022, Clifford neural layers for pde modeling: arXiv preprint arXiv:2209.04934.
[2] Li, Z., Kovachki, N., Azizzadenesheli, K., Liu, B., Bhattacharya, K., Stuart, A., and Anandkumar, A., 2020, Fourier neural operator for parametric partial differential equations: arXiv preprint arXiv:2010.08895.
[3] Wei, W., and Fu, L.-Y., 2022, Small-data-driven fast seismic simulations for complex media using physics-informed fourier neural operators: Geophysics, 87, No. 6, T435–T446