

A discontinuous Galerkin fast sweeping method for Eikonal equation on unstructured triangular meshes

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Summary

Accurate numerical computation of seismic traveltimes for complex geological settings is essential in many seismic applications, such as tomography and migration. Solving the eikonal equation is one of the most commonly used methods for obtaining traveltimes, but currently, most eikonal solvers are based on rectangular grids and are unsuitable for irregular interfaces. To deal with irregular topography and subsurface interfaces, we use the discontinuous Galerkin method (DG) to solve the eikonal equation on unstructured triangular meshes. Compared to rectangular meshes, triangular meshes show higher accuracy in handling complex geometries. The DG solver can obtain high-order accurate traveltimes on a more compact template by utilizing appropriate basis functions.

At present, most triangle-based DG methods solve the equation over the whole domain using a time stepping approach, which incurs significant computational costs. Li et al. (2008) integrated the DG solver in the fast sweeping method (FSM) for rectangular meshes to efficiently achieve second-order accurate solutions. In the article, we apply such FSM acceleration to the DG-based eikonal solver (FSM-DG) for triangular mesh to improve the computational efficiency of traveltimes. To ensure the effective implementation of Gauss-Seidel iterative method, we adopt the ordering strategies based on reference points and l^p -distances proposed by Qian et al. (2007). In general, the iterative methods need a good initial guess for convergence, and FSM-DG is no exception. The finite-difference type fast sweeping method (FSM-FD) proposed by Chen et al. (2024) is used to provide a good initial guess for FSM-DG. Numerical tests demonstrate that the FSM-DG method can solve the eikonal equation with second-order accuracy and obtain more accurate traveltimes on triangular meshes.

Theory

The 2D eikonal equation is given by:

$$\sqrt{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2} = s(x, z), \quad (1)$$

where T denotes the traveltimes, $s(x, z)$ is the slowness. The complex geological area is partitioned into n elements using unstructured triangular meshes, and sorting the triangular meshes based on their l^p -metrics relative to multiple reference points. Then, the Gauss-Seidel iteration method is used to update the traveltimes on the triangular mesh sequentially, where the traveltimes are obtained by solving equation (1) using DG formulation.

Following Cheng and Wang (2014) and Le et al. (2018) to obtain the weak formulation of equation (1), the linear polynomials P_i are chosen for any element K and find $T_h \in \{v: v|_K \in P(K)\}$, such that

$$\begin{aligned} & \int_K H(\nabla_x T_h, \mathbf{x}) \varphi d\mathbf{x} + \int_{\partial K} \min(\tilde{H}_{\mathbf{n}_K}(\nabla_x T_h, \mathbf{x}), 0) [T_h](\mathbf{x}) \varphi^-(\mathbf{x}) ds \\ & - C \frac{\Delta K}{\Delta S_K} \int_{\partial K} \left(S_{\mathbf{n}_K}(\nabla_x T_h, \mathbf{x}) - \left| \tilde{H}_{\mathbf{n}_K}(\nabla_x T_h, \mathbf{x}) \right| \right) [\nabla_x T_h \cdot \mathbf{n}_K](\mathbf{x}) \varphi^-(\mathbf{x}) ds \\ & - 2C \frac{\Delta K}{\Delta S_{\bar{K}}} \int_{\partial \bar{K}} \min(\tilde{H}_{\mathbf{n}_{\bar{K}}}(\nabla_x T_h, \mathbf{x}), 0) (\nabla_x T_h^- \cdot \mathbf{n}_{\bar{K}}) \varphi^-(\mathbf{x}) ds = 0, \quad \forall \varphi \in P(K), \end{aligned} \quad (2)$$

where ΔK is size of the element K , ΔS_K and $\Delta S_{\bar{K}}$ represent the lengths of the edges S_K and $S_{\bar{K}}$. The S_K represents the internal edge shared by two triangles in the computational domain, while $S_{\bar{K}}$ lies on the boundary of computational domain. \mathbf{n}_K is the outward unit normal to the mesh boundary and \mathbf{t}_K the tangential unit vector. At the mesh interface, $[T_h](\mathbf{x}) = T_h^+(\mathbf{x}) - T_h^-(\mathbf{x})$, $\bar{T}_h(\mathbf{x}) = \frac{1}{2}(T_h^+(\mathbf{x}) + T_h^-(\mathbf{x}))$, and $T_h^\pm(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} T_h(\mathbf{x} \pm \varepsilon \mathbf{n}_K)$.

In the DG scheme, the intermediate variables are defined as:

$$\begin{aligned} H(\nabla_x T, x) &= \sqrt{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2} - s(x, z), H_{\mathbf{n}_K} = \nabla_{\nabla T} H \cdot \mathbf{n}_K, \\ \tilde{H}_{\mathbf{n}_K} &= \begin{cases} \frac{H((\nabla_x T_h \cdot \mathbf{n}_K)^+, \overline{\nabla_x T_h \cdot \mathbf{t}_K}, \mathbf{x}^+) - H((\nabla_x T_h \cdot \mathbf{n}_K)^-, \overline{\nabla_x T_h \cdot \mathbf{t}_K}, \mathbf{x}^-)}{[\nabla_x T_h \cdot \mathbf{n}_K](\mathbf{x})}, & [\nabla_x T_h \cdot \mathbf{n}_K](\mathbf{x}) \neq 0 \\ \frac{1}{2} [H_{\mathbf{n}_K}((\nabla_x T_h \cdot \mathbf{n}_K)^+, \overline{\nabla_x T_h \cdot \mathbf{t}_K}, \mathbf{x}^+) + H_{\mathbf{n}_K}((\nabla_x T_h \cdot \mathbf{n}_K)^-, \overline{\nabla_x T_h \cdot \mathbf{t}_K}, \mathbf{x}^-)], & \text{otherwise} \end{cases} \\ S_{\mathbf{n}_K} &= \max\left(\left|\tilde{H}_{\mathbf{n}_K}\right|, \tilde{H}_{\mathbf{n}_K} - H_{\mathbf{n}_K}((\nabla_x T_h \cdot \mathbf{n}_K)^-, \overline{\nabla_x T_h \cdot \mathbf{t}_K}, \mathbf{x}^-), H_{\mathbf{n}_K}((\nabla_x T_h \cdot \mathbf{n}_K)^+, \overline{\nabla_x T_h \cdot \mathbf{t}_K}, \mathbf{x}^+) - \tilde{H}_{\mathbf{n}_K}\right), \end{aligned}$$

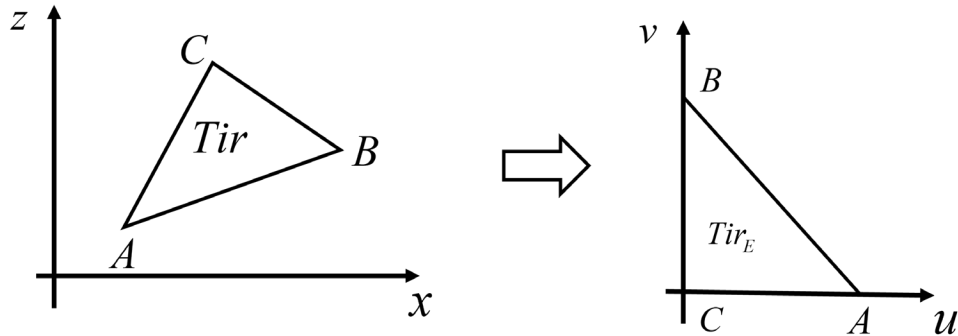


Fig 1. Coordinate transformation from physical triangle to the reference triangle.

Equation 2 involves edge and volume integrals. Therefore, for DG method, it is necessary to map any triangular meshes to the reference triangular mesh and then perform the integral computations. The transformation process is shown in Figure 1. The numerical approximation of T_h can be linearly combined by a set of coefficients and basis functions: $T_h = a\varphi_1 + b\varphi_2 + c\varphi_3$, where $\varphi_1 = 1, \varphi_2 = \sqrt{2}(-1 + 3u), \varphi_3 = \sqrt{6}(-1 + u + 2v)$. The coefficients a, b, c are to be determined. When calculating, we can use FSM-FD to calculate the T_h at the node to initialize the coefficients a, b, c , and then use the FSM-DG to update coefficients.

Results

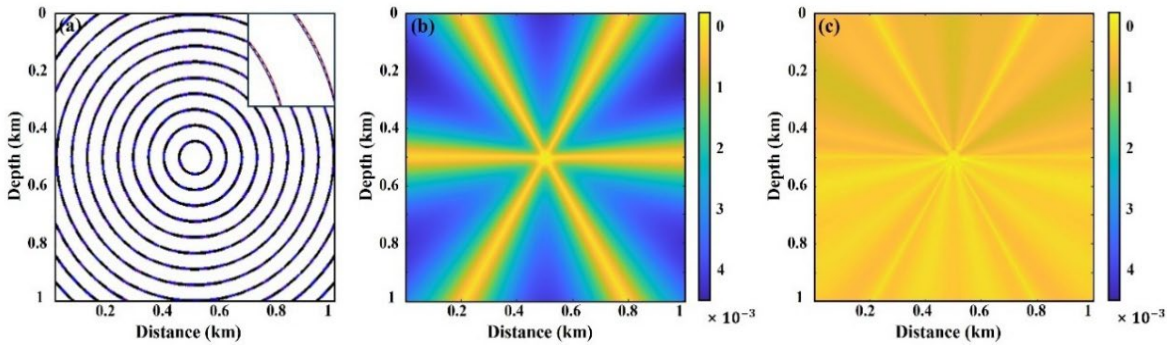


Fig 2. (a) Traveltimes calculated by different methods, and Zoom-in to the corner. Red-curve: exact solution; Blue-curve: FSM-FD; Black-dashed: FSM-DG. Traveltime error obtained by (c) FSM-FD and (d) FSM-DG method to solve the eikonal equation.

Table 1. The numerical error and convergence order of FSM-FD

Side length (m)	L ¹ error	Order	L error	Order
10	2.37×10^{-3}	--	4.51×10^{-3}	--
20	5.17×10^{-3}	1.13	8.99×10^{-3}	0.99
40	1.07×10^{-2}	1.05	2.06×10^{-2}	1.20

Table 2. The numerical error and convergence order of FSM-DG

Side length (m)	L ¹ error	Order	L error	Order
10	4.21×10^{-4}	--	8.51×10^{-4}	--
20	1.52×10^{-3}	1.84	3.03×10^{-3}	1.83
40	7.37×10^{-3}	2.28	1.25×10^{-2}	2.05

In this section, we test the method on a constant velocity model to illustrate its performance. The velocity model is $V = 1000 \text{ m/s}$, and the domain size is $1\text{km}^3 \times 1\text{km}^3 \times 1\text{km}^3$, and we locate the source in the center ($x_s = 0.5\text{km}, y_s = 0.5\text{km}, z_s = 0.5\text{km}$). We discretized the model using the triangular mesh with a side length of 10m, and obtained the traveltimes using FSM-FD and FSM-DG respectively, as shown in Figure 2a. The traveltimes calculated by the FSM-DG (black dashed lines) is closer to the analytical solution (red lines). To demonstrate their differences more intuitively, we obtained the traveltime error calculated by FSM-FD and FSM-

DG, as shown in Figures 2b and 2c. As expected, traveltimes error of FSM-DG is significantly reduced.

We discretize the model using triangular meshes of different sizes, and obtain numerical errors and convergence orders for both methods. From Table 1 and Table 2, we observe first-order convergence in both L^1 error and L^∞ error for FSM-FD, while FSM-DG converges second-order.

To test the validity of the FSM-DG method for complex geological settings, we test the proposed method on the medium with a velocity gradient and an irregular surface, as shown in Figure 2a. The size of this model is $1\text{km} \times 1\text{km}$, and the source is located at $(0.5\text{km}, 0.5\text{km})$. In Figure 2b, we discretize the model using unstructured triangular meshes, and the numbers of meshes and nodes are 9626 and 4947, respectively. Figure 1c shows the traveltimes contours obtained by FSM-FD and FSM-DG, and the two methods can effectively solve traveltimes in complex triangular meshes. In contrast, the numerical solution based on the FSM-DG method is closer to the analytical solution, and the results are more satisfactory.

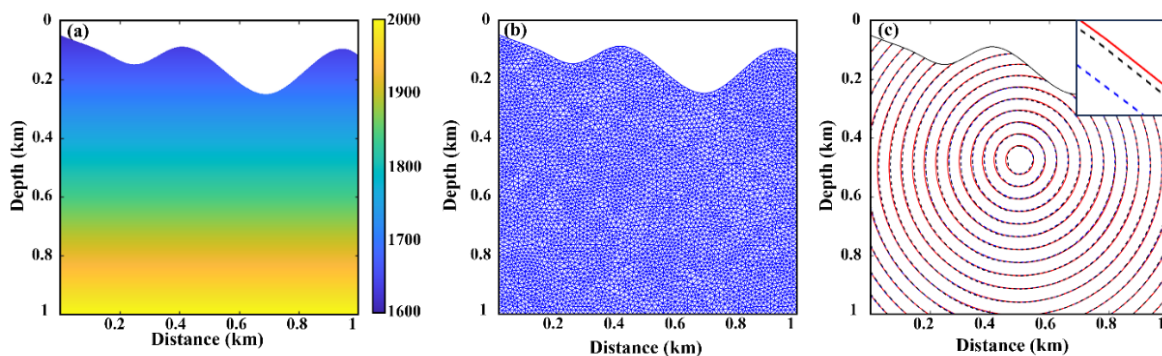


Fig 2. (a) The irregular surface model. (b) the triangulation for the model. (c) Traveltime contours obtained by different methods, and Zoom-in to the corner. Red lines: exact solution; Blue-dashed lines: FSM-FD; Black-dashed lines: FSM-DG.

Conclusions

We apply the fast-sweeping method to a DG-based eikonal solver to obtain seismic wave traveltimes in complex structures. The new method uses the DG solver to solve the eikonal equation in unstructured triangular meshes, which can achieve higher accuracy traveltimes solutions compared to the FD operator. At the same time, the method updates the solution on the meshes sequentially within the FSM framework, which is more efficient than the time-marching method. Numerical examples verify that the method can solve the eikonal equation with second order accuracy, and achieve high-precision seismic wave traveltimes in complex structures.

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